Dialectical proof: should we teach it to physics students?

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ABSTRACT

This paper attempts to introduce a dialectical conception of proof into the discussion of proof teaching. Such a conception is used to design and interpret results of a teaching experience with physics students where an explaining proof of L'Hôpital's rule that unifies both cases, $0/0$ *and* ∞/∞*, was tested.*

1. For a dialectical conception of proof: circularity and essence

It is generally accepted that proof was born in the fifth century B. C. in Greece. Was it accidental that it was born at the same place and time as democracy and banking credit? Perhaps not. A century earlier, tradition held that, in spite of soil impoverishment, the eldest son should stay at home, taking care of the rituals around the sacred fire and guarding his ancestors' tomb. It was the time of reforms of Solon and Dracon. The younger brothers were sent abroad to commerce; they came back rich and increased the power of the *polis* as opposed to that of landowners. The conflict between tradition and money could not be solved by the sword since it stemmed from inside the families. Democracy became necessary. The *agora*, philosophy and, following the same line, mathematical proof emerged. Three centuries later, when it became necessary to diffuse this genial solution to the whole world, the Greeks built Alexandria's light-house, more a symbol than an useful device. Proofs were arranged in logical packets. It was the time of Euclid.

<u>The dialectical circularity</u>

The year 1976 saw two landmarks for Mathematics Education: the announcement of the computer proof of the four color theorem [Apple & Haken, 1976] and the publication of *Proofs and Refutations* [Lakatos, 1976]. Philosophers of mathematics were thrown into turmoil. Is mathematics fallible? Are computers reliable? Will the computer culture introduce new paradigms of proof into mathematics? At the apex of the stir, Horgan [1993] blatantly announced *The death of proof*. While philosophers of mathematics struggled to reestablish peace, mathematics educators discussed the (new) role of proof in their classroom. This story is nicely told in Hanna [1995, 1996].

From Euclid to Hilbert, proof underwent a long development. Instead of departing from objects and common sense truths about them, objects became "symbolic entities which owe their existence only to the fact that they satisfy the rules by which they are axiomatically linked" [Hanna & Jahnke, 1993, p. 425]. With computers, axiomatics linkage further escaped control, as in zero-knowledge and holographic proofs, [see Hanna, 1996, p. 23].

In the course of the discussion stirred up by the computer issue, every item that had traditionally been evoked to present proof as a guarantee of truth was challenged. Hanna [1983, quoted in Neubrand 1989] makes an effort to characterize conditions by which mathematicians accept a new theorem. The new result should *be understandable, significant*, and *consistent*, the author should have an *unimpeachable reputation*, and there should exist a *convincing argument*. Of these five factors, Neubrand stresses the last one: "It is somewhat like a *sine qua non* condition and should therefore head all the other social factors" [Neubrand, 1989, p. 6]. David Hersh reinforces the "convincing" factor and introduces the community of "judges". "In mathematical practice, in the real life of living mathematicians, proof is *convincing argument, as judged by qualified judges*" [Hersh, 1993, p. 389]. "At the stage of creation, proofs are often presented in front of a blackboard, hopefully and tentatively" [ibid. p. 390].

These arguments stress the conception of mathematics as a *discursive practice* [McBride, 1989]. "With few exceptions, mathematicians have only one way to test or "prove" their work – invite everybody who is interested to have a shot at it. So the day-to-day mathematical meaning of "proof " agrees with the colloquial meaning" [Hersh, 1993, p. 392]. One could be tempted to say that proof is just a way of speaking, a form of speech of a certain community: "What mathematicians at large sanction and accept is correct" [Hersh, 1993, p. 392]. Pushing such a shift towards a pragmatic view a little further, we would infer that mathematics is an office room conspiracy of scientists. Scared by this conclusion, we would go back in search of a new and stronger normative attitude. In order to stop swinging back and forth, we only have to assume the dialectical circularity in its sharpest form: *a theorem is true because mathematicians say it is; but they would not say it is true if it were not.* Now we can *move* on.

<u>The movement of the concept: essence and *dasein*</u>

"The movement is the double process and unfolding of the whole; thus each moment places the other at the same time and each one has both moments in itself as two aspects; taken together, these aspects constitute the whole, insofar as they dissolve themselves and make themselves moments of such a whole" [Hegel, 1941, p. 36, my translation].

Throughout the literature about proofs, there are at least three consensual points: 1) proofs have to do with the general idea of *truth* (convincing, explaining, justifying, demonstrating, deducing, etc.); 2) we should certainly teach proofs; 3) formalization is not the best way to teach proofs.

From the perspective of the person who is reading them, formal proofs have long been sharply criticized: "What one generally gets in print is a daunting cliff that only an experienced mountaineer might attempt to scale and even then only with special equipment" [Epstein & Levy, 1995, p. 670].

"The proof follows a course that starts at an arbitrary point, so that one cannot know the relation between this initial point and the result that must come from it. The proof's bearing requires such determinations and such relations and discards others, so that one cannot immediately realize under which necessity this happens; an <u>exterior</u> <u>finality</u> commands such a movement" [Hegel, 1941, p. 37, underlining added].

As a consequence of such criticism, attempts have been made to distinguish aspects of formal proofs capable of providing alternative approaches to be used in classrooms. Many categories have been proposed:

Explanation, *proof* and *demonstration* [Balachef 1987], *preformal* versus formal proofs [Blum & Kirsch, 1991], *proofs that prove* versus *proofs that explain* [Hanna, 1995], *formal* versus *intuitive* proofs [Fishbein, 1982], *proofs to try and test* versus *proofs to establish beyond doubt* [Epstein & Levy, 1995], *analytical* versus *substantial arguments* [Godino & Recio, 1997], *structural* versus *linear-styled* proofs [Alibert & Thomas, 1994], *analytical*, *empirical* and *external* proof schemes [Harel & Sowder, 1996], *humanist* versus *absolutist* mathematics teacher [Hersh, 1993], *technical* versus *critical* perspectives [Garnica, 1995].

All these attempts point to a subjacent aspect of proofs that lies beside or underneath the pure statement of a theorem and its final written form. This aspect hinges on the above-mentioned *discursive practice* that we can now identify with what Hegel calls "exterior finality". In the following paragraphs, I shall underline the specific references to the exterior finality that rules the development of proof:

"Should we give the impression that the best mathematician is some sort of magic conjured out of thin air by extraordinary people when it is actually the result of hard work and of intuition built on the study of many special cases?" [Epstein & Levy, 1995, p. 670].

"(...) a 'convincing argument' is not simply a sequence of correct answers. One always expects some 'qualitative' reason or an intuitive capable basic idea behind the – nevertheless necessary – single steps of the proof" [Neubrand, 1989, p. 4].

"The feeling of the universal necessity of a certain property is not reducible to a pure conceptual format. It is a feeling of agreement, a basis of belief, an intuition – but which is congruent with the corresponding formal acceptance" [Fishbein, 1982, p. 17].

"The best proof is one which also helps mathematicians to understand the meaning of the theorem being proved: to see not only *that* it is true but also *why* it is true" [Hanna, 1995, p. 47].

"The concept of formal (...) proof can become an effective instrument for reasoning process if, and only if, it gets the qualities required by adaptive empirical behavior" [Fishbein, 1982, p. 17].

"(...) the general plan is never revealed (...) and the student may be reduced to merely checking the validity of the deduction at each step" [Alibert and Thomas, 1994, p. 222].

What do these authors mean by "general plan", "adaptive behavior", "feeling of agreement", etc.? What are they pointing at? What are they looking for? Hegel would bluntly call it *the essence*. The "formal proof" from which they are trying to distance themselves, Hegel would call *dasein* ("l'être-là").

"Also in philosophical knowledge, the development of *dasein* is different from the development of the essence or of the inner nature of the thing" [Hegel, 1941, p. 37] .

What Lakatos [1976] describes is the development of essence. In brief, if we assume the dialectical principle that all determinations are relative to each other, we can consider the conceptual movement started in 1976 as the development of a single whole, along which distinct aspects of proof are separated from each other. While we think of proof as a fixed pivot around which we have been turning, proof is actually constantly becoming everything that we have been saying about it.

2. The study

The study was carried out in a one-year freshmen calculus course for physics students at UNESP, Rio Claro, SP, Brazil during 1995-97. Approximate numbers for each year have been: 60 students enroll (40 freshmen plus 20 repeaters), 40 attend classes, 20 pass. Half of the students have part-time jobs and eighty per cent live in nearby cities. The Campus remains empty during weekends. The syllabus covers the first volume of the textbook: Swokowski [1983].

Early in 1995 the Physics Department made the following request of teachers in charge of mathematics courses for the physics students: "We need mathematics as instrumentation for physics. We would like the students to gain familiarity with the textbook. We would like you to teach *less theorems and proofs* and more exercises and applications to physics". A negative reference was made to the linear algebra course where questions like "show that $x.0 = 0$" used to appear in the exams.

What should I have done ? Should I have ignored the request and assumed that my mission would be to open a window through which students would have the opportunity to contemplate the mathematical world during some time, before they proceed in their curriculum? Should I teach proofs in such a context? As far as I know, the question of teaching proofs in mathematics courses for service departments has not been addressed in the literature. Specifically, *should we teach mathematical proofs to physics students?* If so, why and how? This question is not trivial since, as we have seen, mathematical proofs have to do with mathematical truth while physics students are being trained to abide by criteria of truth specific to their science. "If you believe, as many do, that proof is math and math is proof, then, in a math course, you're duty bound to prove something" [Hersh, 1993, p. 396]. Well, I do not believe so. As the new calculus teacher for physics freshmen in 1995, I took the physics department's request as a challenge, not as an interference.

Here is what we [1] did . During 1995, we followed suggestions of Alibert & Thomas 1994, Arsac et al [1992, and Legrand, 1990]. We tried to introduce scientific debates into the classroom. We started with graphical problems about kinematics and tried to gradually introduce mathematical instrumentation as problems became more algebraic. Students were exhorted not to use formulas or results unless they could justify them. Exercises were taken from the textbook and proofs were introduced through worksheets according to the belief that "the main function of proof in mathematics education is surely that of explanation" [Hanna, 1995, p. 47] and that the mathematics teacher should "use the most enlightening proof, not necessarily the most general or the shortest" [Hersh, 1993, p. 397]. The approximately forty students were generally organized in groups of four. Slow learners were invited to extra tutorial sessions once a week. [See Baldino, 1997].

We never succeeded in keeping more than one fourth of the class interested in the debate. In the beginning of next year, the Physics Department complained that the students that we had passed on to them were poor calculators of integrals. Therefore, during 1996 and 1997, the course was split: four hours a week were dedicated to concepts and applications and two hours a week were dedicated to straightforward calculations of limits, derivatives and primitives. Scientific debate was restricted to tutorial sessions. Now the students were told that they would eventually receive worksheets with justifications of the results that they were already using, such as the chain rule and the fundamental theorem of calculus.

L'Hôpital's rule emerged spontaneously from the classroom culture, introduced by those repeating the course. Proofs that we found in the literature could not be classified as explaining proofs. Besides, the case ∞/∞ cannot be immediately reduced to the case 0/0, unless the existence of the quotient limit of the functions can be granted beforehand. Having to show the existence of this limit makes the proof considerably harder. So, we decided to take

[1] Myself and two graduate students: Tania Cabral and Ronaldo Melo.

up Hanna's challenge: "Unfortunately there is no guarantee that every theorem we might like to use will have a proof that explains" [Hanna, 1995, p. 48

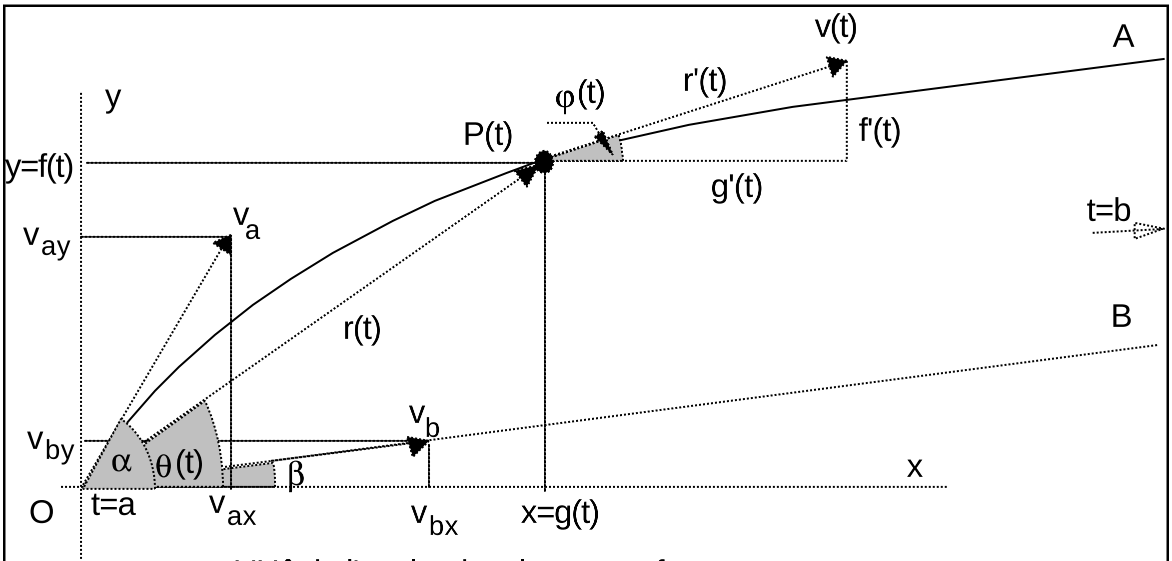
Textbooks suggested in the course's syllabus adopt different strategies in order to circumvent the difficulty with the ∞/∞ case. Hughes-Hallet et al. [1994]] do not even mention L'Hôpital's rule. The authors evoke graphical calculators to solve classical limits and prove the error expression in Taylor's formula by successive integration. Other books do not mention that this case is more difficult: (Carvalho e Silva [1994, p. 279], Ayres Jr. [1981]). Most authors mention the case ∞/∞ but omit the proof and send the reader to "more advanced texts": Linch et al. [1973, p. 514], Apostol [1976, p. 300], Leithold [1976, p. 510], Keisler [1986, p. 246], Swokowski [1983, p. 622]. Shenk [1979, p. 304] leaves the proof to an honors exercise, Strang [1991, p. 153] gives a proof assuming the existence of the limit, and Simmons [1985, p. 569] gives a hint for the proof. Spivak [1967, p. 186], Piskunov [1977, p. 149] and Seeley [1968, p. 643] are among the few that present the proof in detail; unfortunately these are epsilontic proofs.

Seeley also offers one figure. We evaluated that this figure contained the essence of the argument and took it as a starting point to design a worksheet appropriate for physics students. In so doing, we were guided by directives summarized in the following table of oppositions.

Deductive proof	*Dialectical proof*
Development of *dasein*	Development of essence
Hypothesis - thesis – demonstration	Thesis – demonstration - hypothesis
Linearity of the significant chain	Network of models
General case first	Particular case first
Primacy of concept definition	Primacy of concept image

3. Worksheet: Why does L'Hôpital's rule work?

Consider a moving particle along the trajectory AO in the xy-plane. Suppose that at instant t, the particle is at P(t) = (x(t), y(t)) with position vector r(t), as in the figure.



L'Hôpital's rule: development of essence

Suppose that at instant t = a the particle is at the origin and as t tends to b the particle gets away from the origin so that both its coordinates tend to infinity. Mathematically we are saying that: $\lim_{t\to a+} \frac{f(t)}{g(t)} = \frac{0}{0}$ and $\lim_{t\to b-} \frac{f(t)}{g(t)} = \frac{+\infty}{+\infty}$. Let the particle's velocity at instant t be v(t) = r'(t) = (g'(t), f'(t)). Suppose that at t= a the velocity has an initial value v_a and that, as t tends to b, the velocity tends to a final value v_b parallel to the straight line OB. Let $v_a = (v_{ax}, v_{ay})$ and $v_b = (v_{bx}, v_{by})$ be the components of the initial and final velocities. Let the angles φ(t), θ(t), α and β be as in the figure.

1. Describe the trajectory as t tends to b.
2. Fill in the blanks:

$$\lim_{t\to a+} \varphi(t) = \ldots \quad \lim_{t\to a+} \theta(t) = \ldots \quad \lim_{t\to b\_} \varphi(t) = \ldots \quad \lim_{t\to b\_} \theta(t) = \ldots$$

3. Consider the slopes of the straight lines determined by the vectors r'(t), r(t), v_a e v_b and fill in the blanks:

$$\lim_{t\to a+} \frac{f'(t)}{g'(t)} = \ldots \quad \lim_{t\to a+} \frac{f(t)}{g(t)} = \ldots \quad \lim_{t\to b-} \frac{f'(t)}{g'(t)} = \ldots \quad \lim_{t\to b-} \frac{f(t)}{g(t)} = \ldots$$

4. Conclude: why does L'Hôpital's rule work?

Two other sheets with different and higher degrees of formalization were distributed to the students together with this one.

4. Outcomes and discussion

The worksheet was introduced to the students early in June and repeated in early November. In each case we asked for a report: "Explain why L'Hôpital's rule works". Of course, I expected the students to say *Ah ha! Now I know why I am calculating limits in this way.* Interestingly enough, the hard point in formal proofs did not seem to hinder them: all groups could describe reasonably well that, as t tends to b, the trajectory tends asymptotically to a straight line parallel to OB. The existence of the quotient limit of the functions was proved *in action*. Only one group needed help, and that was supplied by hand-waving and dragging an eraser on the table.

I made an attempt to analyze the students' protocols in terms of *proof schemes* proposed by Harel & Sowder [1996]. It seems clear that the proposed explaining proof may be classified as a *transformational proof scheme*: "justifications attend to the generality aspects of a conjecture and involve mental operations that are goal oriented and attended-anticipatory" [ibid. p. 62]. Many protocols clearly indicate an *authoritarian proof scheme.* For these students, L'Hôpital's rule authorizes them to use a procedure either to get rid of the indetermination or to proceed when one gets stuck. "It holds because calculating the quotient of the functions is the same as calculating the quotient of their derivatives". Other protocols indicate a *symbolic proof scheme*: indeterminations are puzzling objects that possess a life of their own, and L'Hôpital's rule explains exactly who such entities are: "When we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, this means that the limit of $\frac{f}{g}$ is equal to the limit of $\frac{f'}{g'}$". Other protocols evoke examples and may be included as *empirical proof schemes*: "We apply L'Hôpital's rule to impose continuity on the indetermination, as in $\frac{\sin x}{x}$".

However, not a single group was able to clearly reproduce the unifying argument: slopes of secant and tangent lines tend to the same value. So, how should we rate our didactical effort? Flat failure? Can we say that we wasted four hours of class-time? Perhaps not, if we look more closely at some protocols such as this one:

> "L'Hôpital's rule holds because, when we apply the quotient rule and we find an indeterminate form, we apply L'Hôpital's rule that proves that, given a point P, when such a point tends to zero, f'(t) and g'(t) tend to the initial values v_{ax} and v_{ay}, where the angle φ tends to the value of α."

That is: "L'Hôpital's rule holds because it proves a property about trajectories". This seems to nicely reproduce, at the level of pedagogy, what Hanna & Jahnke call "appeal to the future" at the level of history. Like in the case of Newton's derivation of Kepler's laws from the Law of gravitation, "that which is proved serves to legitimize the assumptions from which it is derived" [Hanna & Jahnke, 1993, p. 428].

Other protocols are quite difficult to interpret from the point of view of deductive proof. "L'Hôpital's rule holds because, when we apply the limit in indeterminate forms, the function tends to different "angles". However, the slope, when it tends to zero, is the same. Deriving the function, we raise the indetermination and we can find the limits if they exist." A perfect salad! However, if we accept dialectical circularities and obscurities, we may look at this student's development as if we were looking to a developing photograph in a dark room: the picture appears evenly all over the cardboard, not from top to bottom or from left to right. Good mathematicians also "develop" themselves in this way. If a calculus student talks about "limits of infinitesimals" we would take it as a symptom of confused ideas. Nevertheless: "The determination of the tangent to the curve is reduced to the determination of the limit of the ratio of two infinitely small quantities." [Duhamell, 1874, p. 91].

As a final word, we would say that reduction of proof to *deductive proof* is a one-sided view. It may be a necessary ideology for mathematicians' daily scientific practice of theorem-proving, but it does not suffice for mathematics education. "Dialectical view of proof" is an expression borrowed from Hanna & Jahnke, [1993, p. 422]. However, the concept can be traced back to Hegel. *Dialectical proof* is a concept intended to apprehend the development of History and of human subjects in a single unity: the *movement of concept*.

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