Simultaneously assessing intended, implemented and attained conceptions about the gradient [1]

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Abstract

The research question guiding this study was: What are the conceptions about gradient effectively ***attained*** *in a specific calculus course, how are they* ***implemented*** *and how are they related to the conceptions that the curriculum* ***intends*** *to convey? A group of four graduate and two undergraduate students decided to investigate and modify their own mathematical conceptions formed and being formed in calculus and analysis courses. One teacher agreed to conduct tutorial meetings, whose reports were taken as research data. The conclusion was that the* attained *students' conceptions are related to the* intended *curriculum conceptions in a fragmented manner. The conceptions' fragmentation appear to be* implemented *by the demand imposed on the students by the use of a text book, whose analysis indicated that an obsession with loosing control over neglected infinitesimals, ends up hiding the essence of the physical and geometric properties of the gradient under the heavy mantle of mathematical rigor.*

Research question and theoretical references

In late 1995, a group of four graduate students in mathematics education showed dissatisfaction with their mathematics conceptions formed along undergraduate calculus and analysis courses. They decided to investigate and overhaul these conceptions. One teacher accepted to coordinate the group. Two undergraduate (sophomore) students joined in. The group realized the possibility of investigating simultaneously the *intended*, the *implemented* and the *attained* curriculum (Robitaille and Dirks, 1982). The subject of gradient was chosen, since it was the topic being addressed in the calculus course in which the undergraduate students were enrolled at the time. A research question as in the abstract of this article was agreed upon.

The meetings aimed at understanding what was known by the students about gradient, and how it was coming to be organized in the simultaneous calculus course.

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The meetings focused "on conditions under which students will choose to modify, reject, or extend their conceptions" (Confrey, 1990, p. 22). Strike, Hewson and Gertzog (1982, reference ibid. p. 22) "require that a student be dissatisfied with an existing conception and find a new conception intelligible, plausible and fruitful", in order that accommodation can occur. In the case of this research, dissatisfaction with existing conceptions was granted from the beginning and the other conditions were supplied along the way. According to Piaget (cited in Confrey 1995, p. 4) "knowing an object does not mean to copy it - it means acting on it". It can be inferred that, in order to know student's conceptions, it is necessary to act on them, producing changes through a teaching practice. Such was the guide-line of this research. For the final remarks we drew on psychoanalytical theory (Lacan).

Research procedure

A weekly forum of discussion and reflection was started. In our initial planning meeting, it became clear that, whether the participants had seen *gradient* three or more years ago, or whether they had seen it that morning, their spontaneous conceptions[8], centered in the *belief-statement "*vector of partial derivatives*"* would not enable them to sustain a dialogue about such questions as: "What is it used for? What does it mean? What is the geometric representation? Give an example.*"* In brief, there was no other *sense* for *gradient* than the strict mathematical *meaning* of the definition. They did not "remember" any physical or geometrical meaning associated with this word.. To understand how this "forgetting" occurs, it seemed important to investigate the context (classroom, exams, textbook, teacher, etc.) in which the meanings are negotiated during the functioning of the Calculus II course and the subsequent Mathematical Analysis courses. This was partially done and is reported bellow.

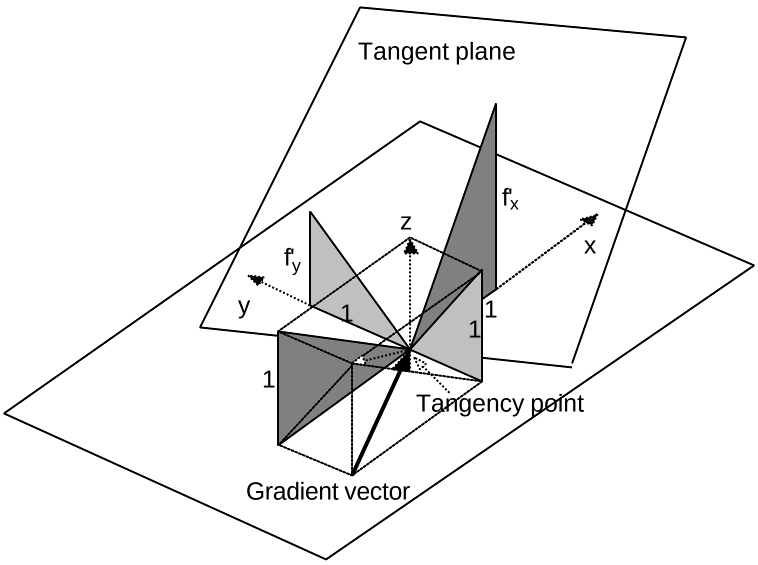
At the beginning of each session, the participants carried out the exercise of talking about what had occurred in the last session; that is to say, the entire process was reviewed in the form of a synthesis. The coordinator intervened to point out the mathematical lapses in the discourse of the person who was speaking, posing questions. At any given moment, someone would be unable to contain themselves and go up to the blackboard to justify what they were saying. When that did not occur, someone was designated by the group to fulfill this role. When someone grasped a point that was being made before the others did, that person was asked to guide the others through questioning in the same manner as the coordinator.

A diary was kept of the first sessions that was presented to the coordinator who made some editing comments. These comments were discussed by the group, with the aim of producing a final script in the form of an article. A fragment of this diary is reprinted below. The teacher's comments, in brackets, reproduce part of his personal notes, taken during and after the sessions. Throughout this process, there was an

[8] "Spontaneous conception" refer to student's conceptions before the research started. For the uses of the term "conception" see Confrey (1990).

ongoing attempt at teaching guided by the mathematical concepts themselves; however, the teaching practice was always subordinate to the learning. At no time did the coordinator push beyond what those present were able to comprehend, as evidenced by the justifications they made, even when this tactic lead to long periods of silence. The following basic rule was always upheld: If you do not know, explain; if you do know, ask questions. Only on the occasions of synthesis did the person who knew explain. A psychoanalysis inspired principle was followed: it is through speaking that one learns and through listening that one teaches.

The attempts at justification outlined by the students led the group to draw the perspective of graphs of real functions with two variables, especially the design of the paraboloid, which functioned as a model for all the properties of the gradient. The difficulty in recognizing the points and curves in the perspective, always precariously drawn on the blackboard, led the coordinator to construct a model, initially improvised using the hard cover of a notebook but, the following week, constructed from acrylic boards, representing the coordinate planes, the tangent plane and lines, the normal and the gradient. From the moment this model was detached from the paraboloid, it became possible to enunciate and justify the properties of the gradient as beliefs of the group, applying only elementary geometric reasoning to the model: the gradient is the projection of a lower normal to the tangent plane on the xy plane, is a vector that is perpendicular to the level curves and has length equal to the largest growth rate of the function at the point. The question then came up: Where does the textbook (Guidorizzi, 1986[9]) state these properties, and how does it proceed to justify them? Why had they escaped the students in their careful page-by-page reading during the Calculus course? The participants then began by locating the statements about these properties in the textbook and examining the book in retrospect to know what steps the author took in an attempt to justify them.



Commented report of first meeting: the paraboloid (August 21st , 1995)[10]

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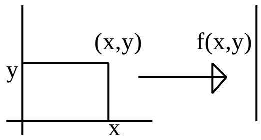
[9] In fact, this is a Brazilian author, but similarity of Calculus textbooks indicate that any other choice would probably have led to the same results.

[10] Taken from the diary of two undergraduate students.

The teacher asked if we had seen the gradient in Calculus II. We responded that we had. *What is it, the gradient?*, he asked. We wrote $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. *Give an example*, he requested. After some reluctance, we proposed the function $f(x,y) = x^2 + y^2$. *What does this function do?*, he asked. The question surprised us. *What do you mean, what does it "do"?* we replied. *Well! Functions function, they have a domain and a counter-domain. How does that one work?* We responded that we had no doubts that it was a paraboloid. *Then draw it*, he said.

[They did not return to me the conception of function as a correspondence. A diagram like this one on the side does not appear to form part of these students' spontaneous conceptions about functions. It would not help to "explain" or "talk" about this since, throughout their schooling, these student must have heard these "speeches" about "correspondences" innumerable times. I prefer to continue and come back to this point using the graph of the paraboloid as a point of departure.]



The teacher suggested that we mark the point (3, 4, 25) and think about the graph of the function.

[They are not seeing the link between the drawing and the points they marked.]

The following definition of graph came up:

$$Gf = \left\{(x,y,z) \in \Re^3 | z = f(x,y); (x,y) \in \Re^2\right\}$$

We finally made explicit the set of level curves.

[It has been a lot of work for me to get them to read the definition as "the set of points (x,y,z) such that…". Remark added later: It is interesting to note that, in the report, there is a reference to something else: to the "set of level curves", not to level curves as sets of points. That would have demanded they use the concept of inverse image, and that would demand that they think of functions as correspondences.]

The teacher asked us to point out the values assumed by the function at the points of the curve of "level one". He emphasized: *the curve of <u>level</u> <u>one</u>*. At that moment we had doubts since, even though we had written the definition of graph, we were not using it. Upon analyzing c_1 we perceived that a function assumes constant values at all points of the curve $x^2 + y^2 = 1$. By the end of the session, the definition of level curve was clear to everyone.

[We did not go back to the gradient; the drawing was never corrected with respect to parallelism; the point (3,4,25) was never located as the intersection of the paraboloid with the planes x = 3 and y = 4.]

[The function $x^2 + y^2$, which should be an example for studying the gradient, became a problem in itself. There was no discussion of a correspondence, but of a "paraboloid". The design of the graph, which should lead to recognition of the domain and counter-domain, became another problem. The sketch of the ellipses inscribed in parallelograms, necessary for the perspective, would have led to the problem of graphic representation of tangents, which I preferred not to get into. The marking of points in the horse-back perspective, which should have aided the sketching of the graph, turned into a time consuming problem of careful measuring. Resorting to the level curves to fix the graph did not work. The level curves were yet another problem. In the meantime, the definition of the graph of a function was evoked in a surprisingly immediate fashion. It was remembered (but not used) and generated a new problem: What does the definition have to do with the "mug"? The memories associated with the equation of the circumference, as a function of two variables, did not lead them to put the level curve on the plane, much less to evoke a cylinder in space.]

[Throughout, the imagination had the model of the mug as a reference. The paraboloid, as a geometrical object, occupied the first plane of attention. "The paraboloid? Sure, we don't have any doubts about it. We know what it is. Axes? Oh, yeah. You have to design axes on the paraboloid. You have to mark the points on the paraboloid. You have to design the level curves of the paraboloid. Finally, you need to connect the definition of "graph of a function" with the paraboloid." One sees that, in the absence of the conception of function as correspondence, which should be the function as a **principle**, the spontaneous conceptions are generated by nucleation of **models**; in this case, the paraboloid.]

[The nature of the fragmentation of the spontaneous conceptions is now somewhat more clear. There is a nuclear model that the student grasps onto and to which the schemes that are stored in different "files" refer, ready to be used there, and only there. In the absence of this model, the "file manager" is lost, and he/she only "remembers having seen it".]

[It is conjectured that it is not the student who is fragmented, and that the problem is neither understood nor resolved with considerations of cognition only. The fragmentation is in response to a specific demand of the university which is fragmented into required courses, optional courses, and exams. In the next course, the paraboloid will no longer be considered; therefore, all that was hanging on it will be without a control center. The "file manager" will be lost. "Why learn another way of marking points on the graph if we already know how it looks like? Why learn to draw inscribed ellipses in a parallelogram if there are no exams in this course?" they seemed to say.]

Commented report of the analysis of the textbook (November 5-19, 1995)[11]

Through our thoughts and reasoning about the acrylic model, and using only the resources of elementary geometry, everything seemed very clear to us. We had summarized our understanding with the following points:

- the gradient is the vector of partial derivatives (definition)
- the gradient is perpendicular to the level curves
- the gradient points in the direction of the greatest growth of the function
- the length of the gradient is the rate of growth of the function in its direction

Not one of these properties seemed unusual to us. Now we were certain that we had already heard about them, that we had seen them in classes and earlier readings. We were curious to know how these properties that looked so familiar to us, were discussed and demonstrated in calculus books like the one used as a textbook in the course in which two of us were enrolled.

[11] Taken from the report of two graduate students in Mathematics Education.

We started by locating the perpendicularity statement and went searching backwards in order to find out how it was demonstrated and on what previous results the author made it to it depend. We looked for the essential nature of the proposition hoping for the transparency that we had obtained through our reasoning about the model. We found that the corresponding material in the textbook is complicated; it consists of a series of premises and statements that led us astray when we tried to trace back the theoretical path leading to the properties. We concluded that it was necessary to read the book from the first page. These fundamental statements appear "en passant" and, to justify them, the author lays out a virtual arsenal of theorems, including the chain rule, internal products, and even, unnecessarily, the implicit functions theorem! Finally we found a text of U. D'Ambrósio about the *mystification of knowledge* that best expressed what we felt. We can affirm that, in our study group, the phenomenon of demystification of knowledge occurred in a very clear form. It was actually a study of ethnomathematics.]

"Rarely does anyone argue that the origin of knowledge resides in the people and obeys a very specific socio-cultural context. The explanations given for this knowledge are naturally partial and at times it appears with an apparent lack of coherence and comes impregnated with a strong *mysticism.* This knowledge generated by the people passes through a process of structuring and coding that, afterwards, is expropriated by powerful groups. In this way, this same knowledge (...) originating in the people becomes accessible to them only in a structured, coded form, most of the time subjected to *mystification* that results in institutional processes of devolution, such as schools, professions, academic grades, and the whole series of training mechanisms. The executors of the devolution to the people of these diverse bodies of knowledge should be recognized by the same power structure, in such a way as to secure their ideological commitment. This credentialing occurs by way of a system of filters; the individual normally loses sight of the process by which they are being co-opted and which goes from the mystical, normally present in the origin of knowledge, to the mystified, as though this same knowledge presents itself to be dressed in a system of codes" (D'AmbrÛsio, 1989).

[The future teachers, as social agents, should offer guarantees, just as the textbook guarantees all of the steps presented therein. Nonetheless, it is necessary that they "lose sight of the process", precisely to better exercise their guaranty function. For this, the book was perfect.]

[The operation of [12] separating from the surface of the paraboloid, the plane and the tangent lines made concrete and tangible by the model, did not impede the students. From the moment we substituted the surface by a plane and the coordinate curves by tangent lines, Euclidean geometry took care of the rest and everything became clear. On the other hand, the author seems determined to keep under vigilance the infinitesimals that are to be overlooked, making explicit the errors that tend to zero when divided by their respective increments. It appears that in this book, as in most calculus books, a true <u>obsession</u> <u>with</u> <u>loss</u> guides the entire presentation and ends up hiding the essence of the physical and geometric properties of the gradient under the heavy mantle of a premature control of the mathematical rigor. In vain,

12 This is basically an operation of overlooking the second order infinitesmals.

the students try to *make sense* of what they read. They can only follow the book line by line and check the rigor of the mathematical *meaning*. The organization of didactic books is, after all, one of the factors that allows us to explain the fragmented appearance of spontaneous conceptions. The obsession with the control of the overlooked infinitesimal institutes an *accountability of loss* that takes precedence over learning.]

Final words: some psychoanalytical inspired remarks

Initially, some of the participants demonstrated a relationship with mathematics somewhere between *afflictive* and *fuzzy*. As they have already graduated from college and could be doing something else with their time, for one reason or another, they chose to be there and prolong or relive this old relationship; in this we should recognize a repetition that, for Lacan, is "the sister of *juissance*". It has to do, then, with a *symptom*. A *transferential* (affective?) *relation* installed itself at that point that made it possible for these participants to enter into the experience of learning. It was expected that the coordinator would occupy the position of speaker, the subject-supposed-to-know.

The desire of the participants was, certainly, sustained in fantasies which we did not intend to delve into, except for those which were undeniably present in the process. As to what lay beyond *selective listening*, the "*fantasy of omnipotence through the domination of knowledge*" (Walkerdine, 1988, Chap. 9) was present *"- Now it is so clear. Why did we not learn this in this way?"* exclaimed the students. The symptom brought by the image of the severe father, the one that awakens the hysterical, revealed itself during the process, when one of the participants verbalized *"- I keep looking for responses to give to him (the coordinator)! Actually I don't have to do that."* She sat down but a bit later she was back at the blackboard. This same symptom revealed itself again at the end, in the report of a dream of another participant who went to a party at the moment a third one was struggling to resolve a problem that was necessary to write this article: *"- I dreamed I was at a party and he (the coordinator) was there looking very angry. He was sitting on a sofa with a computer in front of him, writing . . ".* At the beginning, clearly, everyone thought they should know the answers to the questions that were asked. Later they began to verbalize that not knowing was irrelevant. Finally the "I know/I don't know" problematic disappeared. Meanwhile, one question remains open. To the point that psychoanalysis projects itself beginning with selective listening, it remains to be seen if this could be the correspondent to what lies beyond fantasy?[13].

The affliction and the fuzziness go on. Although they show comprehension more quickly when asked about the subjects worked on at the beginning, fuzziness is still present when a new subject is addressed. It seems as though they incorporated affliction and fuzziness as their way of being with respect to mathematics. We can say that they identify themselves with affliction and fuzziness, whose memory they have so

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[13] On this point, there is some disagreement among the authors.

much fun with. They no longer appear worried about freeing themselves of these symptoms; on the contrary, we would say they enjoy them. Affliction and fuzziness became the "high" of the group. The student who dreamed about the "severe father" later revealed that *"- I didn't tell you everything."* She recently broke up with her boy-friend of two years, who told her *"- You're not the same person I started out loving."* There is a passage there, forbidden to the teacher, that only the psychoanalyst can transpose. This virus is truly dangerous.

Where would traditional teaching have guided these people, who came pushed by symptoms and images of the severe father and the search for knowledge, reproducing afflictive fuzziness from this same subject of knowledge? Certainly, it would have guided them to one of those "recycling courses", so that their demands would have been attended to before the image of the severe father, disguised as a teacher, who attempts to transmit knowledge by way of explanations, showing himself as a plain subject. Simultaneously, the symptoms of affliction and detachment would be driven away by a loving hoax and by whatever trick at the hour of evaluation, always in the hope of, by having learned, no one would demonstrate fuzziness nor affliction. To the degree to which traditional teaching repeats this scheme, there is a symptom there as well, that calls for interpretation. As a repetition, it has to do with *juissance*, but whose?

Instead of this, we followed the opposite path. Instead of trying to suppress or modify the symptom, we strove for people to identify with it. Interpretation of the symptoms (Cf. Szizek, 1992, Cap.VII).

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